

ON BOUNDED UNIVALENT FUNCTIONS WHOSE RANGES CONTAIN A FIXED DISK

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ABSTRACT. Let \mathcal{S} denote the standard normalized class of regular, univalent functions in $K = K_1 = \{z: |z| < 1\}$. Let \mathcal{F} be a given compact subclass of \mathcal{S} . We consider the following two problems. *Problem 1.* Find $\max|a_2|$ for $f \in \mathcal{F}$. *Problem 2.* For $|z| = r < 1$, find the \max (\min) $|f(z)|$ for $f \in \mathcal{F}$. In this paper we are concerned with the subclass $\mathcal{S}_d^*(M) = \{f \in \mathcal{S}: K_d \subset f(K) \subset K_M\}$. Through the use of the Julia variational formula and the Loewner theory we determine the extremal functions for Problems 1 and 2 for the class $\mathcal{S}_d^*(M)$, for all d, M such that $\frac{1}{4} < d < 1 < M < \infty$.

I. Introduction. Let \mathcal{S} denote the class of functions $f(z) = z + a_2z^2 + \dots$, that are regular and univalent in $K = K_1 = \{z: |z| < 1\}$. Let \mathcal{F} be a given compact subclass of \mathcal{S} . We consider the following two problems.

Problem 1. Find $\max|a_2|$ for $f(z) \in \mathcal{F}$.

Problem 2. For $|z| = r < 1$, find \max (\min) $|f(z)|$ for $f(z) \in \mathcal{F}$.

It is well known [4] that for $f(z)$ in \mathcal{S} , $f(K)$ contains $K_{1/4}$ and the function $k(z) = z/(1 - z)^2$ solves Problems 1 and 2 for the class \mathcal{S} and hence for the subclass \mathcal{S}^* of starlike functions in \mathcal{S} .

In [14] T. Suffridge solved Problem 1 for the class \mathcal{S}_d^* where $\mathcal{S}_d^* = \{f \in \mathcal{S}^*: f(K) \supset K_d\}$. He showed that an extremal domain for Problem 1 for \mathcal{S}_d^* , $\frac{1}{4} < d < 1$, has as its boundary a single arc AC of ∂K_d and two radial rays from A and C to ∞ . In this paper Suffridge also conjectured that an extremal domain for Problem 1 in the class $\mathcal{S}_d = \{f \in \mathcal{S}: f(K) \supset K_d\}$ is the domain having as its boundary a single arc AC of ∂K_d and a radial slit to ∞ from the midpoint of AC . E. Netanyahu [11] verified this conjecture.

In this paper we will be concerned with the subclass $\mathcal{S}_d^*(M)$ of \mathcal{S} , $\mathcal{S}_d^*(M) = \{f \in \mathcal{S}^*: K_d \subset f(K) \subset K_M\}$. If we let $\gamma(M)$ be the radius of the Koebe disk for $\mathcal{S}^*(M)$, i.e., the largest disk with center $w = 0$, contained in $f(K)$ for all $f \in \mathcal{S}^*(M)$, we need only consider $d \geq \gamma(M)$ for the class $\mathcal{S}_d^*(M)$. As is well known, $\gamma(M) \rightarrow \frac{1}{4}$ as $M \rightarrow \infty$, while $\gamma(M) \rightarrow 1$ as $M \rightarrow 1$. Schwarz's lemma shows that for $f \in \mathcal{S}_d^*(M)$, $f(K)$ cannot contain K_d with $d > 1$ and the only function in $\mathcal{S}_1^*(M)$, or $\mathcal{S}_d^*(1)$ is $f(z) = z$.

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In this paper we shall, through the use of variational methods, determine the extremal functions for Problems 1 and 2 for the class $\mathcal{S}_d^*(M)$.

We observe that Suffridge's result follows from ours when $M = \infty$. It is immediate that if the extremal function $f^*(z)$ solving Problem 2 is the same for all r sufficiently close to zero, then $f^*(z)$ also solves Problem 1. As we shall see the extremal function for Problem 2 in $\mathcal{S}_d^*(M)$ has this property and thus our results generalize Suffridge's result.

In §§II and III of this paper we introduce the Julia variational formula and show how it can be utilized to solve Problem 2 for the classes of functions under consideration. In these two sections, we will frequently need to use the concept and properties of the conformal mapping radius of a domain. The conformal mapping radius of a simply connected domain Ω with respect to $w = 0$ (denoted m.r. Ω) is defined as follows: If $w = f(z) = Rz + a_2z^2 + \dots$ maps K , 1-1, onto $\Omega = f(K)$, with $R > 0$, then $R = \text{m.r. } \Omega$.

In §IV we introduce additional variational methods that will be needed to study Problem 2 for the class $\mathcal{S}_d^*(M)$. In particular, we show how to combine the Julia variational formula with the Loewner slit mappings to produce variations within the class $\mathcal{S}_d^*(M)$.

In §V, we combine the variational methods of §IV with those of §III to solve Problem 2 for $\mathcal{S}_d^*(M)$. We prove that the boundary of an extremal domain giving the solution to Problem 2 in $\mathcal{S}_d^*(M)$, $M < \infty$, consists of an arc AC of ∂K_d , two radial sides AB and CD , and the arc BD of ∂K_M connecting B to D .

A routine mapping radius argument as in [1] shows that the extremal domain described for $\mathcal{S}_d^*(M)$ is in fact the range of a function in $\mathcal{S}_d^*(M)$. We let $f^*(z, d, M)$ denote this function. We can determine $f^*(z, d, M)$ implicitly as follows. Consider the function

$$\begin{aligned} F(z) &= zf^*(z)/f^*(z) \\ (1.1) \quad &= \left\{ [a(z + z^{-1}) + b + 2] / [a(z + z^{-1}) + b - 2] \right\}^{1/2}, \\ &F(0) = 1, \end{aligned}$$

where $a \geq b/2 + 1$, $b \geq 0$. $F(z)$ is found by a sequence of elementary transformations mapping K onto the right half-plane $\text{Re}\{w\} > 0$ minus the slits $\{w: 0 \leq w \leq c\}$ and $\{w: w \geq d\}$ where

$$d = [(b + 2a + 2)/(b + 2a - 2)]^{1/2}$$

and

$$c = [(b - 2a + 2)/(b - 2a - 2)]^{1/2}.$$

Since $\text{Re}\{w\} > 0$, $f^*(z)$ is in \mathcal{S}^* .

Let ϕ_1 and ϕ_2 be defined such that $0 < \phi_1 < \phi_2 < \pi$ and $\lim_{r \rightarrow 1} f(re^{i\phi_1}) =$

∞ and $F(e^{i\phi_2}) = F(e^{-i\phi_2}) = 0$. Let $u + iv = F(e^{i\phi})$ ($0 < \phi < 2\pi$). Since

$$u + iv = \frac{\partial \arg f^*(e^{i\phi})}{\partial \phi} - i \frac{\partial \log |f^*(e^{i\phi})|}{\partial \phi}$$

(any branch of $\arg f^*(e^{i\phi})$, $0 < \phi < 2\pi$, may be chosen), it follows from the boundary behavior of $F(z)$ that $f^*(K)$ is the set $\{w: |w| < M, -\alpha < \arg w < \alpha\} \cup \{w: |w| < d, \alpha \leq \arg w \leq 2\pi - \alpha\}$ for some d, M and α where $\alpha = \arg f^*(e^{-i\phi_2})$. Since the power series for $f^*(z)$ has real coefficients, the domain $f(K)$ is symmetric with respect to the real axis.

We have

$$(1.2) \quad f^*(z) = z \exp \left[\int_0^z [F(t) - 1] t^{-1} dt \right]$$

where $F(t)$ is defined by (1.1).

From the previous discussion of the boundary behavior of $F(z)$ it follows that for a fixed d and M , an a and b will be determined by

$$f^*(-1) = -\exp \left[\int_0^{-1} [F(t) - 1] t^{-1} dt \right] = d$$

and

$$f^*(1) = \exp \left[\int_0^1 [F(t) - 1] t^{-1} dt \right] = M.$$

II. Preliminary material. In this section we formulate the Julia variational formula and show how it can be utilized to produce variations used in studying Problems 1 and 2 of §I.

Let $f(z)$ be in \mathfrak{S} and map K onto a domain D whose boundary contains a piecewise analytic curve Γ which has left- and right-hand tangents at each point. Let $n(w)$ be the unit normal to Γ at $w \in \Gamma$. Let $\phi(w)$ be a real-valued, continuous, piecewise differentiable function on Γ which vanishes at the nonanalytic points of Γ . If $\varepsilon > 0$ is sufficiently small, $\hat{w} = w + \varepsilon \phi(w)n(w)$ maps Γ , 1-1, onto a curve $\hat{\Gamma}$, and $\hat{\Gamma} \cup \partial \hat{D}/\Gamma$ is the boundary of a domain \hat{D} . Julia [7] proved that if $\hat{f}(z)$ maps K , 1-1, onto \hat{D} with $\hat{f}(0) = 0$, then $\hat{f}(z)$ is given by

$$(2.1) \quad \hat{f}(z) = f(z) + \frac{\varepsilon z f'(z)}{2\pi i} \int_{\Gamma} \frac{\xi + z}{\xi - z} \frac{\phi(w)n(w)}{[\xi f'(\xi)]^2} dw + o(\varepsilon),$$

where $o(\varepsilon)$ denotes a term analytic in z , $z \in K$, such that $o(\varepsilon)/\varepsilon$ converges uniformly to 0 for z in compact subsets of K . Here $w = f(\xi)$, for $\xi = e^{i\theta}$ on the arc of $|\xi| = 1$ that corresponds to Γ under $f(\xi)$.

Let

$$(2.2) \quad d\Psi = \frac{n(w)\phi(w)dw}{i[\xi f'(\xi)]^2} = \frac{\phi(w)d\theta}{|\xi f'(\xi)|}.$$

Then $d\Psi$ is real and has the sign of $\phi(w)$,

$$(2.3) \quad \hat{f}(z) = f(z) + \frac{\varepsilon z f'(z)}{2\pi} \int_{\Gamma} \frac{\xi + z}{\xi - z} d\Psi + o(\varepsilon)$$

and

$$(2.4) \quad \hat{f}'(0) = f'(0) + \frac{\varepsilon}{2\pi} \int_{\Gamma} d\Psi + o(\varepsilon).$$

Thus, we have from (2.3) that

$$\log[\hat{f}(z)/z] = \log[f(z)/z] + \frac{\varepsilon}{2\pi} \int_{\Gamma} \frac{zf'(z)}{f(z)} \frac{\xi + z}{\xi - z} d\Psi + o(\varepsilon).$$

Hence

$$(2.5) \quad \operatorname{Re} \left\{ \left[\log \frac{\hat{f}(z)}{z} \right] \right\} - \operatorname{Re} \left\{ \left[\log \frac{f(z)}{z} \right] \right\} = \frac{\varepsilon}{2\pi} \int_{\Gamma} \sigma(w) d\Psi + o(\varepsilon),$$

where

$$(2.6) \quad \sigma(w) = \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \frac{\xi + z}{\xi - z} \right\},$$

$w = f(\xi)$ and $\xi = e^{i\theta}$. Thus

$$h(\xi) = \frac{zf'(z)}{f(z)} \frac{\xi + z}{\xi - z}$$

maps $|\xi| = 1$ onto a circle for fixed z , $|z| < 1$. Consequently $\operatorname{Re}\{h(\xi)\}$ varies in a certain monotonic manner as ξ moves on $|\xi| = 1$, i.e., there exists two points $e^{i\theta_0}$ and $e^{i\theta_1}$, $0 \leq \theta_0 < \theta_1 < \theta_0 + 2\pi$, on $|\xi| = 1$ separating $|\xi| = 1$ into two arcs γ_1 and γ_2 such that $\operatorname{Re}\{h(\xi)\}$ decreases strictly as ξ moves on each of the two arcs γ_1 and γ_2 from $e^{i\theta_0}$ to $e^{i\theta_1}$. The function $\operatorname{Re}\{h(\xi)\}$ attains its extreme values at the points $e^{i\theta_0}$ and $e^{i\theta_1}$.

We will now outline the method in which equation (2.5) and the monotonicity of $h(\xi)$ can be used to solve Problems 1 and 2. The basic idea is due to Krzyz [9].

First consider a domain with boundary Γ containing three analytic arcs that are images of three arcs of $|\xi| = 1$ under $f(\xi)$. It follows from the monotonic behavior of $\operatorname{Re}\{h(\xi)\}$, that two of the three arcs, denoted by l_1 and l_2 satisfy

$$(2.7) \quad \min_{w \in l_1} \sigma(w) \geq \max_{w \in l_2} \sigma(w).$$

In the sequel (2.7) will be referred to as *the monotonic property of $\sigma(w)$* .

If we consider the two arcs l_1 and l_2 where (2.7) holds, we can choose a $\phi(w) > 0$ and a $\phi(w) < 0$ on the open arcs l_1 and l_2 , respectively, and $\phi(w) = 0$ on the remainder of the boundary such that

$$(2.8) \quad \int_{l_1} \frac{\phi(w)n(w)}{i[\xi f'(\xi)]^2} dw = \int_{l_2} \frac{[-\phi(w)]n(w)}{i[\xi f'(\xi)]^2} dw.$$

Inequality (2.7), and (2.2), in view of the monotonicity property of $\sigma(w)$, imply

$$(2.9) \quad \int_{l_1} \frac{\sigma(w)\phi(w)n(w)}{i[\xi f'(\xi)]^2} dw > \int_{l_2} \frac{\sigma(w)[- \phi(w)]n(w)}{i[\xi f'(\xi)]^2} dw.$$

It follows from (2.4) that, when (2.8) holds, $\hat{f}'(0) = f'(0)$, and when (2.9) holds, $\operatorname{Re}\{\log(\hat{f}(z)/z)\} > \operatorname{Re}\{\log(f(z)/z)\}$, where $z \in K$ is fixed. If $\hat{f}(z)$ belongs to a given class of functions containing $f(z)$, and if such a construction is possible, the function $f(z)$ clearly cannot yield a solution to Problem 2 for the given class in the maximum case.

Similarly, putting $f_1(z) = -f(-z)$ we obtain a function $\hat{f}(z)$ such that $\operatorname{Re}\{\log(\hat{f}(z)/z)\} < \operatorname{Re}\{\log(f(z)/z)\}$, where $z \in K$ is fixed. Thus whenever this construction is possible and $\hat{f}(z)$ belongs to a given class of functions containing $f(z)$, the function $f(z)$ evidently cannot yield a minimal value to the functional $\operatorname{Re}\{\log(f(z)/z)\}$ for the given class.

III. Boundary variations. Let \mathfrak{B} denote the class of all domains that are the range of functions in \mathfrak{S} and whose boundaries lie in the closure of the annulus S . Let $\Omega \in \mathfrak{B}$ with boundary Γ . We will show how to obtain a new domain in \mathfrak{B} by considering certain combinations of the following three types of variations of a linear segment contained in Γ . Let DA , AB and BC be any three adjacent arcs of Γ , with AB a straight line segment, and denote by γ the remainder of Γ . We shall define:

VARIATION I. An inward rotation of AB about an endpoint, say A , whenever AB is not tangent to ∂K_d and BC is either

- (a) a straight line segment, or
- (b) an arc of ∂K_M or ∂K_d .

VARIATION II. An outward rotation of AB about an endpoint, say A , whenever either

- (a) $B \notin \partial K_M$ or ∂K_d and BC is a straight line segment, or
- (b) $B \in \partial K_M$ (∂K_d) and BC is either a straight line segment or an arc of ∂K_M (∂K_d).

To define Variation I(a) when AB and BC are straight sides we let λ equal the interior angle at ABC and consider four cases:

$$(3.1) \quad \begin{array}{ll} \text{(i)} \quad 0 < \lambda \leq \pi/2, & \text{(ii)} \quad \pi/2 < \lambda < \pi, \\ \text{(iii)} \quad \pi < \lambda < 3\pi/2, & \text{(iv)} \quad 3\pi/2 \leq \lambda < 2\pi. \end{array}$$

We rotate the side AB inward by moving B either along the side BC to $B' \in BC$ in cases (i) and (ii), or, along the extensions of the side BC to B' on this extension in cases (iii) and (iv). In all four cases we then use the Julia variational formula (2.1) to obtain the mapping function, $f_1(z)$, for the domain Ω' , bounded by $\gamma DAB'C$. Since Ω' is not obtained from Ω by a normal displacement in all cases, we will need an intermediate step to obtain the desired variational formula for $f_1(z)$, i.e., we first make a normal displacement of $\partial\Omega$ obtaining a domain Ω'' , and the corresponding mapping function, $f_2(z)$ directly from (2.1). We then show that the correction term $f_1(z) - f_2(z)$ is $o(\varepsilon)$ uniformly on compact subsets of K .

In case (i) we draw through B the inward perpendicular BB'' to AB where B'' is chosen such that $AB'' \cap K_d$ is empty. The side AB'' meets BC at B' . In order to obtain a normal displacement we rotate side BB'' inward to BB_1'' , where B_1'' lies on AB' and angle ABB_1'' is less than angle ABB' . Denote angle ABB_1'' by β . Let B_1 be the perpendicular projection of B_1'' onto AB . Denote angle B_1AB_1'' by α . We define

$$(3.2) \quad \varepsilon\phi(w) = \begin{cases} -\varepsilon|w - A| \tan \alpha, & w \in AB_1, \\ -|w - B| \tan \beta, & w \in B_1B, \\ 0, & \text{otherwise.} \end{cases}$$

Let Ω'' be the varying domain defined by the normal displacement $\varepsilon\phi(w)n(w)$ of $\partial\Omega$. Let $f_2(z)$ be the corresponding mapping function such that $f_2(K) = \Omega''$. The Julia variational formula may be applied, giving

$$(3.3) \quad f_2(z) - f(z) = \frac{\varepsilon z f'(z)}{2\pi i} \int_{AB} \frac{\xi + z}{\xi - z} \frac{\phi(w)n(w)}{[\xi f'(\xi)]^2} dw + o(\varepsilon).$$

We denote by Ω' , the varying domain with boundary $\gamma DAB'C$, and let $f_1(z)$ be its corresponding mapping function such that $f(K) = \Omega'$. We will show

$$(3.4) \quad f_2(z) - f_1(z) = o(\varepsilon).$$

Let $g_2(w, w_0)$ and $g_1(w, w_0)$ be the Green's functions for Ω'' and Ω' , respectively. Then, since Ω'' can be obtained by a suitable normal displacement of $\partial\Omega'$ (using a finite number of steps if necessary) we have, using an argument similar to that of Krzyz [9], $g_2(w, 0) - g_1(w, 0) = o(\varepsilon)$. (3.4) then follows by a straightforward argument utilizing the fact that $g_i(w, 0) = -\log|f_i^{-1}(w)|$, $i = 1, 2$.

Thus by comparing the corresponding mapping functions of Ω' and Ω we have

$$(3.5) \quad f_1(z) - f(z) = \frac{\varepsilon z f'(z)}{2\pi i} \int_{AB} \frac{\xi + z}{\xi - z} \frac{\phi(w)n(w)}{[\xi f'(\xi)]^2} dw + o(\varepsilon)$$

where $\phi(w)$ is defined by (3.2).

In case (ii) we draw through B the inward perpendicular BB'' to AB where B'' is chosen such that $AB'' \cap K_d$ is empty. The prolongation of AB'' meets BC at B' . In order to obtain a normal displacement we rotate BB'' inward to BB_1'' until the angle $BB_1''B'$ is less than $\pi/2$. Denote the angle ABB_1'' by β . Let B_1 be the perpendicular projection of B_1'' onto AB . Let $\alpha = \text{angle } B_1AB_1''$. Then $\phi(w)$ is defined by (3.2) and Ω'' , $f_2(z)$, Ω' , and $f_1(z)$ are also defined as in case (i). Equation (3.5) follows using the same argument as in case (i).

In case (iii) equation (3.5) can be obtained directly from (2.1) without the need of a correction term as in (3.4). Case (iv) follows similarly to case (ii).

We now define Variation I(b). The varied domain will be obtained by rotating the side AB such that B moves along either the arc BC of ∂K_M to $B' \in BC$ or the extension along ∂K_d of an arc BC of ∂K_d to B' on this extension. In either case, proceeding as in Variation I(a), the variable domain Ω' arises by deleting from Ω the curvilinear triangle ABB' . Then the derivation of equation (3.5) follows as in I(a) except that the straight line segment BB' is replaced by an arc BB' of ∂K_M or ∂K_d .

REMARKS. 1. If Ω is a polygonal domain then Variation I(a) preserves the number of vertices of the boundary of Ω .

2. If Variation I is applied to a nonradial segment of the boundary of a starlike domain in \mathfrak{B} , the varied domain is again starlike.

To define Variation II(a), where AB and BC are straight line segments with $B \notin \partial K_M$ or ∂K_d , we consider the four cases of $\lambda = \text{angle } ABC$ as in (3.1), but we denote these cases as (i'), (ii'), (iii'), and (iv'), respectively. We will rotate the side AB outward by moving B either along an extension of the side BC to B' on this extension in cases (i') and (ii') or along the side BC to $B' \in BC$ in cases (iii') and (iv'). By using the two step procedure, as in Variation I, we show that equation (3.5) holds in this variation with $\phi(w)$ appropriately defined. We give the details of case (i'), noting its similarity to case (ii). We observe that case (ii'), case (iii'), and case (iv') are similar to case (iii), case (iv), and case (i) respectively.

In case (i') we draw BB'' —the outward perpendicular to AB at B —in such a way that the prolongation of BC meets the prolongation of BB'' at B' where B' does not lie outside ∂K_M . Rotate BB' inward to BB_1'' until angle $BB_1''B'$ is less than $\pi/2$. Let β be the angle ABB_1'' which is less than $\pi/2$. Let B_1 be the perpendicular projection of B_1'' onto AB . Denote angle B_1AB_1'' by α . Then, we can define

$$\varepsilon\phi(w) = \begin{cases} \varepsilon|w - A| \tan \alpha, & w \in AB_1, \\ |w - B| \tan \beta, & w \in B_1B, \\ 0, & \text{otherwise.} \end{cases}$$

The variable domain Ω'' will be determined by the normal displacement $\varepsilon\phi(w)n(w)$ of the boundary of Ω , whereas Ω' is the varying domain with the

boundary $\gamma DAB'C$. Comparing the corresponding mapping functions for Ω and Ω' we obtain as in the previous cases, equation (3.5).

We define Variation II(b) where the endpoint B of side AB is in ∂K_M (or ∂K_d) and BC is an arc of ∂K_M (or ∂K_d) or a straight line segment.

If BC is an arc of ∂K_M (∂K_d), then the argument proceeds as in case (ii') [(iii')], except that the line BB' is replaced by an arc BB' of ∂K_M (∂K_d). The variable domain Ω' arises by adjoining to Ω the curvilinear triangle ABB ; with variable domain Ω' arises by adjoining to Ω the curvilinear triangle ABB' ; with variable B' on ∂K_M (∂K_d). If BC is a straight line segment, it is clear that if we obtain the variable domain Ω' by adjoining to Ω the curvilinear triangle ABB' with variable B' remaining on ∂K_M (∂K_d), the proof of (3.5) proceeds as in case (ii') [(iii')].

We will use combinations of the above described variations in the following manner.

LEMMA 3.1. *Let $f(z)$ be in a given subclass \mathfrak{T} of \mathfrak{S} . Let Γ denote the boundary of $f(K)$, where Γ is contained in the closure of the annulus $S = \{w: d < |w| < M\}$. Suppose Γ contains two line segments l_1 and l_2 that satisfy inequality (2.7). If l_1 can be rotated outward as in Variation II while l_2 is rotated inward, as in Variation I, in such a way that the new mapping function $\hat{f}(z)$ produced is also in \mathfrak{T} , then the function $f(z)$ cannot give a solution to Problem 2 for the given class \mathfrak{T} .*

PROOF. The proof follows from the discussion in §II and, in particular, inequality (2.9).

IV. Boundary variation of a slit mapping. In preparation for considering Problem 2 for the class $\mathfrak{S}_d^*(M)$ we will need to define a variation of a different type than those considered in §III.

Suppose $\Omega = f(K)$, $f \in \mathfrak{S}_d^*(M)$, and $\Gamma = \partial\Omega$ is a curvilinear polygon. Assume Γ contains a linear slit of Ω , i.e., a radial segment each "side" of which is a boundary arc of Ω . Let $w_0 = f(e^{i\theta_0})$ be the tip of the slit T . Following Loewner [10] we define

VARIATION III. *The shortening of a linear slit.*

We can parametrize T by $w = w(t)$, $0 \leq t \leq t_0$, $w(0) = w_0$, and denote by $w = g(z, t)$, $g(0, t) = 0$, $g'(0, t) > 0$, the function that maps K onto $\Omega = f(K)$ minus the segment $[w_0, w(t)]$. We denote the domain $g(K, t)$ by Ω_t . It follows from Loewner's theory that under these conditions $w = g(z, t) = e^t z + \dots$ satisfies the Loewner equation

$$(4.1) \quad \frac{\partial g}{\partial t} = z \frac{1 + k(t)z}{1 - k(t)z} \frac{\partial g}{\partial z}$$

where $g(z, 0) = f(z)$ and $k(t)$ is a continuous function with $|k(t)| = 1$ and

$\lambda(t) = 1/k(t)$ is the point on $|z| = 1$ whose image under $g(z, t)$ is the tip $w(t)$ of the shortened slit.

In order to utilize Variation III in solving Problem 2 for $\mathcal{S}_d^*(M)$ we will combine Variation III with Variation I of §III for functions $f \in \mathcal{S}_d^*(M)$. We then show that if a certain relationship exists between the shortened slit and the rotated side, the function $f(z)$ cannot yield a solution to Problem 2 within $\mathcal{S}_d^*(M)$.

First we note that since $\partial\Omega_t$ is a curvilinear polygon, $g(z, t)$ has a continuous extension to $|z| = 1$.

Since $g(z, t) = e^t z + \dots$, $g'(0, t) > 1$. To obtain a function that is normalized, we consider the following: If there exists a side AB of $\partial\Omega_t$ that can be rotated inward as in Variation I of §III, then the Julia variational formula (2.1) defines a function $\hat{g}(z, t, \varepsilon)$ that maps K onto a domain with the same boundary as Ω_t except that the side AB has been rotated inward appropriately. For convenience we define $\phi(w)$ so that $\phi(w) > 0$ on AB and use $-\varepsilon$ for $\varepsilon > 0$ in (3.5). We obtain from equation (3.5)

$$(4.2) \quad \hat{g}(z, t, \varepsilon) = g(z, t) - \frac{\varepsilon z g'(z, t)}{2\pi} \int_{AB} \frac{\xi + z}{\xi - z} d\Psi + o(\varepsilon),$$

where $g'(z, t) = \partial g / \partial z$ and, as in (2.2),

$$d\Psi = \phi(w) n(w) dw / i [\xi g'(\xi)]^2$$

is real and positive for $\phi(w) > 0$. From (4.2) we obtain

$$\hat{g}(z, t, \varepsilon) = e^t \left[1 - \frac{\varepsilon}{2\pi} \int_{AB} d\Psi + o(\varepsilon) \right] z + \dots$$

To normalize \hat{g} we define $t = t(\varepsilon)$ by

$$(4.3) \quad e^t \left[1 - \frac{\varepsilon}{2\pi} \int_{AB} d\Psi + o(\varepsilon) \right] = 1;$$

i.e.,

$$t = -\log \left[1 - \frac{\varepsilon}{2\pi} \int_{AB} d\Psi + o(\varepsilon) \right].$$

It follows that

$$(4.4) \quad \left. \frac{\partial t}{\partial \varepsilon} \right|_{\varepsilon=0} = \frac{1}{2\pi} \int_{AB} d\Psi.$$

When $z = z_0$ in (2.6) we obtain

$$(4.5) \quad \sigma_1(w) = \operatorname{Re} \left\{ \frac{z_0 f'(z_0)}{f(z_0)} \frac{\xi + z_0}{\xi - z_0} \right\}.$$

The main result of this section is

LEMMA 4.1. Let $f(z) \in \mathcal{S}_d^*(M)$, $f(K) = \Omega$, $\Gamma = \partial\Omega$, and $|f(z)| \leq |f(z_0)|$

for $|z| = r$, $0 < r < 1$. Suppose Γ contains a radial slit T of Ω . Let l_1^* be the image under $f(z)$ of a sufficiently small arc, containing $e^{i\theta_0}$, of $|\zeta| = 1$, where $f(e^{i\theta_0})$ is the tip of T . Also assume Γ contains a straight line segment l_2 such that l_1^* and l_2 satisfy

$$(4.6) \quad \min_{w \in l_1^*} \sigma_1(w) > \max_{w \in l_2} \sigma_1(w).$$

If the slit T is shortened as in Variation III, while l_2 is rotated inward as in Variation I of §III in such a way that the new mapping function produced is also in $\mathfrak{S}_d^*(M)$, then the function $f(z)$ cannot yield a solution to Problem 2 for $\mathfrak{S}_d^*(M)$.

PROOF. If the variations are performed in such a way that the class $\mathfrak{S}_d^*(M)$ is preserved, we need only show that (4.6) implies that the function \hat{g} , as defined by (4.2), has a greater maximum (minimum) modulus on $|z| = r$ than $f(z)$. Let z_0 , $|z_0| = r$, be the point where the max (min) of $|f(z)|$ occurs. Then when (4.6) holds, we determine the existence of an $\varepsilon_0 > 0$ such that

$$(4.7) \quad |\hat{g}[z_0, t(\varepsilon_0)]| > f(z_0).$$

We do this by showing that (4.6) implies

$$(4.8) \quad \frac{\partial}{\partial \varepsilon} \left[\log \left| \frac{\hat{g}[z_0, t(\varepsilon)]}{z_0} \right| \right] \Big|_{\varepsilon=0} > 0.$$

(4.8) implies the existence of an $\varepsilon_1 > 0$ such that for all $\varepsilon \in [0, \varepsilon_1]$,

$$|\hat{g}[z_0, t(\varepsilon)]| > |\hat{g}(z_0, 0)| = |f(z_0)|.$$

Then, since \hat{g} is normalized for any ε satisfying (4.3) there exists a positive $\varepsilon_0 \in (0, \varepsilon_1)$ such that (4.8) holds.

To prove (4.8), we have from (4.1), (4.2), and (4.4), that

$$\begin{aligned} & \frac{\partial}{\partial \varepsilon} \left[\log \left| \frac{\hat{g}[z_0, t(\varepsilon)]}{z_0} \right| \right] \Big|_{\varepsilon=0} \\ &= \operatorname{Re} \left\{ \frac{1}{\hat{g}[z_0, t(\varepsilon)]} \left[\frac{\partial \hat{g}[z_0, t(\varepsilon)]}{\partial \varepsilon} \right. \right. \\ & \quad \left. \left. - \frac{\partial}{\partial \varepsilon} \left(\frac{\varepsilon z_0 g'[z_0, t(\varepsilon)]}{2\pi} \int_{AB} \frac{\zeta + z_0}{\zeta - z_0} d\Psi + o(\varepsilon) \right) \right] \right\} \Big|_{\varepsilon=0} \end{aligned}$$

$$\begin{aligned}
&= \operatorname{Re} \left\{ \frac{1}{\hat{g}} \left[\frac{\partial}{\partial t} g[z_0, t(\varepsilon)] \frac{\partial t}{\partial \varepsilon} \right. \right. \\
(4.9) \quad &\quad \left. \left. - \varepsilon \cdot \frac{\partial}{\partial t} \left(\frac{z_0 g'[z_0, t(\varepsilon)]}{2\pi} \int_{AB} \frac{\zeta + z_0}{\zeta - z_0} d\Psi + o(1) \right) \cdot \frac{\partial t}{\partial \varepsilon} \right. \right. \\
&\quad \left. \left. - \frac{z_0 g'[z_0, t(\varepsilon)]}{2\pi} \int_{AB} \frac{\zeta + z_0}{\zeta - z_0} d\Psi + o(1) \right] \right\} \Big|_{\varepsilon=0} \\
(4.10) \quad &= \frac{z_0 f'(z_0)}{f(z_0)} \left[\frac{1}{2\pi} \int_{AB} d\Psi \cdot \operatorname{Re} \left\{ \frac{1 + e^{-i\theta_0} z_0}{1 - e^{-i\theta_0} z_0} \right\} \right. \\
&\quad \left. - \frac{1}{2\pi} \int_{AB} \operatorname{Re} \left\{ \frac{\zeta + z_0}{\zeta - z_0} \right\} d\Psi \right],
\end{aligned}$$

provided we justify the existence of

$$\begin{aligned}
(4.11) \quad &\frac{\partial}{\partial t} \left(\frac{z_0 g'[z_0, t(\varepsilon)]}{2\pi} \int_{AB} \frac{\zeta + z_0}{\zeta - z_0} d\Psi \right) \\
&= \frac{\partial}{\partial t} \left[\frac{z_0 g'[z_0, t(\varepsilon)]}{2\pi} \int_{AB} \frac{\zeta + z_0}{\zeta - z_0} \frac{\phi(w)n(w) dw}{i[\zeta g'(\zeta)]^2} \right].
\end{aligned}$$

We observe that (4.10) is positive if

$$(4.12) \quad \operatorname{Re} \left\{ \frac{z_0 f'(z_0)}{f(z_0)} \frac{\zeta_0 + z_0}{\zeta_0 - z_0} \right\} \int_{AB} d\Psi > \int_{AB} \operatorname{Re} \left\{ \frac{z_0 f'(z_0)}{f(z_0)} \frac{\zeta + z_0}{\zeta - z_0} \right\} d\Psi$$

where $\zeta_0 = e^{i\theta_0}$ is the preimage of the tip of the slit T . Since I_2 and I_1^* , which contains the tip $f(e^{i\theta_0})$ of T , satisfy (4.6), (4.12) holds, and (4.8) follows. Thus (4.7) will be proved when we justify the existence of (4.11). We first investigate the integral in (4.11). The existence of the partial of this integral presents no difficulty when A and B are interior points of a straight line segment in Γ . So, assume Γ has a corner at A and the section of Γ adjacent to AB at A is either a straight line segment or an arc of ∂K_M . For the cases when $\phi(w)$ has different left and right derivatives at a point B_1 in the interior of AB we will split the integral into the sum of two integrals, where one integral is taken over AB_1 and the other integral is taken over B_1B . It clearly suffices to justify the existence of the integral in (4.11) over AB_1 . We use the fact [3] that in a neighborhood of a corner, say $g(\zeta_1, t)$, of the boundary of Ω , we have

$$g(\zeta, t) = g(\zeta_1, t) + (\zeta - \zeta_1)^{1-\beta} h(\zeta)$$

where $(1 - \beta)\pi$ is the interior angle at the corner and $h(\zeta)$ is analytic and nonzero in a neighborhood of ζ_1 . Thus,

$$\int_{AB} \frac{\zeta + z}{\zeta - z} \frac{\phi(w)n(w)dw}{i[\zeta g'(\zeta)]^2}$$

can be written with the change of variable $\zeta = \exp\{i \arg[\zeta_1 + (\zeta_2 - \zeta_1)s]\}$ as

$$k \int_0^1 \left| \frac{\zeta + z}{\zeta - z} \right| \frac{(\zeta - \zeta_1)h(\zeta)}{(1 - \beta)h(\zeta) + (\zeta - \zeta_1)h'(\zeta)} \left| \operatorname{Im} \left\{ \frac{\zeta_2 - \zeta_1}{\zeta_1 + (\zeta_2 - \zeta_1)s} \right\} \right| ds \quad (4.13)$$

where k is a constant relative to t and ζ_1 and ζ_2 are the preimages of A and B_1 respectively under $g(\zeta, t)$. We observe that $\zeta = g^{-1}(w)$ satisfies the differential equation

$$(4.14) \quad \frac{\partial \zeta}{\partial t} = \zeta \frac{1 + k(t)\zeta}{1 - k(t)\zeta}.$$

Indeed

$$\frac{\partial \zeta}{\partial t} = \frac{\partial}{\partial w} g^{-1}(w) \cdot \frac{\partial w}{\partial t} = \zeta \frac{1 + k(t)\zeta}{1 - k(t)\zeta}.$$

Let ζ_0 belong to a compact subarc of $|\zeta| = 1$ that does not contain the points $k(t)$, $0 \leq t \leq t_0$. For $z \in K$ arbitrarily close to ζ_0 we have for $w = g(z, t) \in \Omega_t$,

$$(4.15) \quad \frac{\partial z}{\partial t} = \frac{\partial}{\partial w} g^{-1}(w) \cdot \frac{\partial w}{\partial t} = z \frac{1 + k(t)z}{1 - k(t)z}.$$

If we denote the right-hand side of (4.15) by $R(t, \zeta, s) = R$, then it can easily be shown that R is continuous and satisfies a Lipschitz condition with respect to t and that $\partial R / \partial \zeta$ and $\partial R / \partial s$ exist and are continuous for all $s \in [0, 1]$ and ζ in a sufficiently small neighborhood of ζ_0 . Hence by a standard theorem of differential equations [2], it follows that equation (4.15) is also valid for w on a simple arc of $\partial \Omega_t$. Thus a straightforward calculation shows that the partial of the integrand of (4.13), with respect to t exists and is continuous. Thereby we can conclude the existence of (4.11) upon showing that $\partial g'(z, t) / \partial t$ exists. But, since the partial with respect to z of the right-hand side of (4.1) exists and is continuous as a function of z and t , the existence of $\partial g'(z, t) / \partial t$ follows.

We observe that if $f(z)$ is rotated such that $|f(z)| \geq -f(-z_0)$, then, by the appropriate changes in the signs of the above arguments we have the existence of an $\varepsilon_0 > 0$ such that $|\hat{g}[-z_0, t(\varepsilon_0)]| < -f(-z_0)$, for an l_1^* and l_2 satisfying (4.6). This completes the proof of Lemma 4.1.

In studying Problem 2 for $\mathfrak{S}_d^*(M)$ we shall also need to define two modifications of Variations I and II of §III. Using the same assumptions on a straight line segment AB of $\Gamma = \partial f(K)$ as in §III we define

VARIATION I'. *The pushing in of a straight line segment AB at the midpoint C of AB , and*

VARIATION II'. *The pushing out of a straight line segment AB at the midpoint C of AB .*

Variation I' (II') is defined by rotating AC and BC inward (outward) as in Variation I (II) of §III, about A and B respectively. Both AC and CB are rotated α degrees, forming an isosceles triangle with base AB .

REMARK. If Variation I' or Variation II' is applied to a nonradial segment of a starlike domain, and α is sufficiently small, the varied domain is again starlike.

Finally, we need to mention a special case of Variation I used by Hummel [5] and Robertson [13]. We define

VARIATION I''. *The hinging of one side of a slit at a point of the side.*

Denote "one side" of a slit T as AB with A at the tip and B at the other end of the slit. Let $C \in AB$, $C \neq B$. Then the slit is hinged at C if the straight line segment CB is rotated inward as in Variation I of §III.

REMARKS. Assume that $f(z) \in \mathcal{S}_d^*(M)$, $\Gamma = \partial f(K)$, and that Γ contains a radial slit T .

1. If T is shortened as in Variation III, starlikeness is preserved.
2. If Γ is a polygon then shortening T also preserves the number of vertices of Γ .
3. If T is hinged at a point of its side as in Variation I'', then starlikeness is preserved.
4. If T is hinged at the tip of the slit using Variation I(a) of §III in Variation I', then the number of vertices of Γ is preserved.
5. We observe that both Variation I' and Variation I'' can be combined with Variation III to produce new domains in $\mathcal{S}_d^*(M)$.

V. Results for starlike functions.

THEOREM 5.1. *Let $f(z) \in \mathcal{S}_d^*(M)$, $\gamma(M) \leq d \leq 1$, and $M \geq 1$. Then, for $|z| = r$ fixed, $0 < r < 1$, $-f^*(-r) \leq |f(z)| \leq f^*(r)$, where*

$$f^*(z) = f^*(z, d, M)$$

is defined by (1.2).

PROOF. The proof of Theorem 5.1 consists of a series of lemmas.

Since the cases for $d = 1$ and $M = 1$ have already been discussed in §I we will assume $d < 1$ and $M > 1$.

Let \mathcal{Q} denote the class of starlike domains $\Omega = f(K)$ for $f \in \mathcal{S}_d^*(M)$. Let \mathcal{Q}_n denote the subclass of \mathcal{Q} which consists of all domains Ω whose boundary is an n -sided (linear) polygon. We allow radial slits in $\partial\Omega$ and we consider a slit as two "sides" of $\partial\Omega$. Clearly \mathcal{Q} (and \mathcal{Q}_n) and the corresponding mapping

functions form *compact* classes. Thus there exists a domain $\Omega_1^* = f^*(K)$ [$\Omega_n^* = f_n^*(K)$] where $f^*(z)$ [$f_n^*(z)$] satisfies $|f(z)| \leq f^*(z_0)$ [$|f_n(z)| \leq f_n^*(z_0)$], $|z| = r < 1$, for all $f(K) \in \mathcal{Q}$ [$f_n(K) \in \mathcal{Q}_n$].

The following lemma shows that $\cup \mathcal{Q}_n$ is dense in \mathcal{Q} .

LEMMA 5.2. *If $\Omega \in \mathcal{Q}$, then there exists a sequence of domains $\{\Omega_n\}_{n=3}^\infty$ where $\Omega_n \in \mathcal{Q}_n$, such that Ω_n converges to Ω , i.e., Ω is the kernel of the sequence $\{\Omega_n\}$ in the sense of Carathéodory.*

PROOF. It suffices to prove for every $\varepsilon > 0$, there exists a polygonal domain $P_n \in \mathcal{Q}_n$ such that ∂P_n is contained in an ε -neighborhood of $\partial \Omega$.

Let $\varepsilon > 0$. Let $f(z) \in \mathcal{S}_d^*(M)$ be the corresponding mapping function in $\mathcal{S}_d^*(M)$ such that $f(K) = \Omega$. We consider $f_r(z) = f(rz)/r$. It follows from the properties of starlike functions and the maximum and minimum modulus principles that for $0 < r < 1$, $f_r(z) \in \mathcal{S}_d^*(M)$ and $d < |f_r(z)| < M$ on $|z| = 1$. So, for each r , $0 < r < 1$, the image of $|z| = 1$ under $f_r(z)$ is a 1-1, analytic curve, contained in the annulus S and bounds a starlike domain. Also, for r sufficiently close to 1, $\partial f_r(K)$ is contained in an ε -neighborhood of $\partial f(K)$. Let r_0 be sufficiently close to 1 and let $\Gamma = \partial f_{r_0}(K)$.

It is sufficient to prove that we can approximate Γ to within ε by a polygonal line p such that the domain bounded by p is starlike and has mapping radius one.

We divide the arc Γ into two parts Γ_1 and Γ_2 such that Γ_1 consists of all the subarcs of Γ where $\operatorname{Re}\{1 + zf''(z)/f'(z)\} \geq 0$ for $|z| = 1$ and Γ_2 consists of all the subarcs of Γ where $\operatorname{Re}\{1 + zf''(z)/f'(z)\} < 0$, for $|z| = 1$. It is easily shown that each of Γ_1 and Γ_2 contains only a finite number of components. Thus we need only show that each component can be replaced by an appropriate polygonal line.

Let γ_1 be a component of Γ_1 and γ_2 a component of Γ_2 . We can approximate γ_1 (γ_2) from within (without) by connecting points on γ_1 (γ_2) and from without (within) by intersecting segments of tangents to neighboring points on γ_1 (γ_2). By taking points on γ_1 (γ_2) sufficiently close it follows that both methods of approximation produce a polygonal line p having the same endpoints as γ_1 (γ_2) and such that p is contained in an ε -neighborhood of γ_1 (γ_2). Also, because of the nature of the methods of approximation, the domain bounded by $p + \Gamma \setminus \gamma_1$ (γ_2) is starlike.

Hence it suffices to prove that by combining the two methods of approximation the mapping radius can be preserved. Let γ be an arbitrary component of either Γ_1 or Γ_2 . We divide γ into two proper subarcs, say γ_3 and γ_4 . Then we approximate γ_3 from within by a polygonal line p_3 and we approximate γ_4 from without by a polygonal line p_4 . If we denote the polygonal line $p_3 + p_4$ by p , then using the fact that a refinement of either approximation appropriately increases or decreases the mapping radius, it follows that p

can be chosen so that the domain bounded by $p + \Gamma \setminus \gamma$ has mapping radius one. Thus, Lemma 5.2 is proved.

If Ω_n converges to Ω , the Carathéodory Kernel Theorem assures that for the function f_n and f , $f_n(K) = \Omega_n$ and $f(K) = \Omega$, $f_n(z)$ converges to $f(z)$ uniformly on compact subsets of K . Thus, it follows that any convergent subsequence $\{\Omega_{n_k}\}_{k=1}^\infty$ of the extremal domains for Problem 2 in \mathcal{Q}_n converges to a domain $\Omega_1^* \in \mathcal{Q}$, which is an extremal domain for Problem 2 within \mathcal{Q} . In order to characterize Ω_1^* we must first determine certain properties of $\partial\Omega_n^*$.

LEMMA 5.3. *Let Γ_n^* be the boundary of the extremal domain Ω_n^* for Problem 2 within \mathcal{Q}_n . Γ_n^* has at most two radial slits, and all but at most two of the distinct sides (not on slits) of Γ_n^* are either chords of ∂K_M or tangent to ∂K_d .*

PROOF. To show that $\Gamma_n^* = \partial\Omega_n^*$ has not more than two radial slits, we assume that Γ_n^* has at least three radial slits. Since $f(z)$ has a continuous extension to ∂K , $\sigma_1(w)$ can be defined for all $w \in \Gamma_n^*$. Thus, recalling the monotonic property of $\sigma_1(w)$, as defined in (4.5), we let M and m denote the two points in Γ_n^* where $\sigma_1(w)$ attains its maximum and minimum values, respectively. M and m divide Γ_n^* into two arcs such that $\sigma_1(w)$ is strictly increasing as w moves along either arc of Γ_n^* from m to M . At least one of these two arcs contains a radial slit and "a side" of another slit. Clearly, we can choose one of the slits, denoted by T , to define an l_1^* and the "side", denoted by AB , of another slit, which we denote by T' , with tip at A , to act as an l_2 in inequality (4.6). Recall, if $f(e^{i\theta_0})$ is the tip of T , then l^* is the image of a sufficiently small arc, containing $e^{i\theta_0}$, of $|\zeta| = 1$. Now, since we can shorten the slit T as in Variation III, of §IV, and hinge the side AB of slit T' at A as in Variation I" of §IV, preserving starlikeness and the number of vertices in the boundary, we can produce another domain in \mathcal{Q}_n . Thus, Lemma 4.1 implies that Ω_n^* cannot be an extremal domain for Problem 2 within \mathcal{Q}_n . Since this yields a contradiction, Γ_n^* can have no more than two radial slits.

Next, we assume that Γ_n^* has at least three distinct sides (not sides of slits) which are neither chords of ∂K_M nor tangent to ∂K_d . From monotonicity of $\sigma(w)$, one of the sides will act as an l_1 in inequality (2.7), and another side will act as an l_2 in (2.7). Since l_1 is not a chord of ∂K_M , it can be rotated outward, as in Variation II(a) of §III, about one of its endpoints, while preserving starlikeness and without increasing the number of vertices of the boundary. Since l_2 is not tangent to ∂K_d , it can be rotated inward as in Variation I(a) of §III, preserving starlikeness and the number of vertices of the boundary. Hence, by rotating l_1 outward while rotating l_2 inward and taking $\phi(w)$ so that equality holds in (2.8), the new domain obtained will be in \mathcal{Q}_n . Thus, by Lemma 3.1, Ω_n^* cannot be an extremal domain for Problem 2 within \mathcal{Q}_n . This yields a contradiction, and Lemma 5.3 is proved.

We shall need

LEMMA 5.4. *Let $R = \{w: d < r_2 < |w| < r_1 < M\}$. If $\Gamma_n^* = \partial\Omega_n^*$, then there exists, independently of n , a maximum number N of sides of Γ_n^* that intersect \bar{R} .*

PROOF. Let $C_1 = \partial K_{r_1}$ and $C_2 = \partial K_{r_2}$.

We first consider those sides of $\partial\Omega_n^*$ that are chords of ∂K_M . If a chord of ∂K_M intersects \bar{R} , then there exists a minor arc of ∂K_M of fixed minimum arc length, say c_1 , subtended by the chord. The minimum arc length c_1 would be the arc length of the minor arc subtended by a chord of ∂K_M which is tangent to C_1 . Choose an integer k_1 such that $k_1 c_1$ exceeds the circumference of ∂K_M . Then, independently of n , no more than k_1 sides of $\partial\Omega_n^*$, which are chords of ∂K_M , can intersect \bar{R} .

If a ∂K_d -tangent, say l , intersects \bar{R} we let P be the point which is the radial projection of a point where l intersects the circle C_2 . Let P' be the point of tangency of the ∂K_d -tangent l . The arc length of $\widehat{PP'}$ is a fixed length c_2 , depending only on ∂K_d and C_2 . Since starlikeness prevents overlapping of the radial projections of the ∂K_d -tangents on ∂K_d , if k_2 is an integer such that $k_2 c_2$ exceeds the circumference of ∂K_d , then there can be no more than k_2 ∂K_d -tangents that intersect \bar{R} .

Allowing the possibility of two radial slits and two sides of $\partial\Omega_n^*$ that are not ∂K_d -tangents or chords of ∂K_M , it follows that no more than $N = k_1 + k_2 + 6$ sides of $\partial\Omega_n^*$, independent of n , can intersect R , and Lemma 5.4 is proved.

We now begin the characterization of Ω_1^* .

LEMMA 5.5. *If $\Gamma_1^* = \partial\Omega_1^*$, then Γ_1^* has at most four radial slits.*

PROOF. We shall need the following property, denoted by P_1 , of the kernel Ω of a sequence of domains $\{\Omega_n\}_{n=1}^\infty$. Let C be a compact set contained in $\partial\Omega$. For every $\delta > 0$, there exists an $N = N(\delta)$ such that for each $n > N$, if $w \in C$, then there exists a $w_n \in \partial\Omega_n$ such that $|w_n - w| < \delta$. This property follows easily from the definition of the kernel of a sequence of domains.

Now, assume Γ_1^* has at least five radial slits. We draw two circles C_1 and C_2 , centered at the origin with radii r_1 and r_2 , respectively. r_1 and r_2 are determined by requiring that the two points of the intersection of C_1 and C_2 with one of the shortest of five of the radial slits, divides that slit into thirds. Let $\{a_1, a_2, \dots, a_5\}$, $\{b_1, b_2, \dots, b_5\}$ be the points of intersection of these slits with C_1 and C_2 respectively, where a_i and b_i are on the same slit, $i = 1, 2, \dots, 5$. Choose c_i the midpoint of $[a_i, b_i]$. Choose δ so small that no chord or ∂K_d -tangent passing through a δ -neighborhood of an a_i or b_i passes through a δ -neighborhood of c_i for $i = 1, 2, \dots, 5$. Let C be the set of segments $\{[a_i, b_i] | 1 \leq i \leq 5\}$. Then Property P_1 assures that for this δ there exists an N such that a point of $\partial\Omega_n^*$ lies within a δ -neighborhood of each point of C . From starlikeness, the only way the boundary, $\partial\Omega_n^*$, can intersect a δ -neighborhood of a_i , a δ -neighborhood of b_i and a δ -neighborhood of c_i is

for $\partial\Omega_N^*$ to contain at least one side which is neither a chord of ∂K_M nor a ∂K_d -tangent for each $i = 1, 2, \dots, 5$. From the definition of δ this would require $\partial\Omega_N^*$ to have more than two radial slits or more than two distinct sides which are neither chords of ∂K_M nor ∂K_d -tangents. But this implies a contradiction to Lemma 5.3 and Lemma 5.5 is proved.

LEMMA 5.6. *Let $\Gamma_1^* = \partial\Omega_1^*$. Then the intersection of Γ_1^* with any proper subannulus of S is comprised of a finite number of linear segments.*

PROOF. Let $R = \{w: d < r_2 < |w| < r_1 < M\}$ be a proper subannulus of S . We show that $R \cap \Gamma_1^*$ is comprised of linear segments for any $r_1, r_2, d < r_2 < r_1 < M$. If $\Gamma_n^* = \partial\Omega_n^*$ then, from Lemma 5.4 for each n , $\bar{R} \cap \Gamma_n^*$ is a set of at most $N = N(\bar{R})$ line segments, i.e., $\bar{R} \cap \Gamma_n^* = \{[a_{n1}, b_{n1}], [a_{n2}, b_{n2}], \dots, [a_{nN}, b_{nN}]\}$. For an appropriate subsequence $\{n_k\}$, $a_{n_k i}$ and $b_{n_k i}$ converge to a_i and b_i , respectively, for $i = 1, 2, \dots, N$. Hence each point of the line segment $[a_{n_k i}, b_{n_k i}]$ converges to a corresponding point of the segment $[a_i, b_i]$ and $[a_i, b_i] \subset \bar{R}$. Also Ω_n^* converges to Ω_1^* . Since each boundary point of Ω_1^* is a limit point of boundary points of the $\Omega_{n_k}^*$, it follows that $R \cap \Gamma_1^*$ is contained in the linear segments $\{[a_i, b_i] | 1 \leq i \leq N\}$ and Lemma 5.6 is proved.

LEMMA 5.7. *$\Gamma_1^* = \partial\Omega_1^*$ contains no nonradial straight line segments in S and at most two distinct radial sides (not on slits) in S .*

PROOF. Assume AB is a nonradial straight line segment of Γ_1^* . Let A_1B_1, A_2B_2, A_3B_3 , be three distinct closed line segments contained in AB such that $A_iB_i \cap \bar{K}_d = \emptyset$ and A_i and B_i are distinct from A and B for $i = 1, 2, 3$. Then from the monotone property of $\sigma(w)$ in (2.6), one of these subsets of AB , say A_1B_1 , will act as in l_1 in inequality (2.7), while another of these subsets of AB , say A_2B_2 will act as an l_2 in (2.7). We can now push out $l_1 = A_1B_1$ at its midpoint as in Variation II' of §IV, and push in $l_2 = A_2B_2$ at its midpoint as in Variation I' of §IV, taking $\phi(w)$ so that equality holds in (2.8). Then, since AB is nonradial, these variations preserve starlikeness and the varied domain, thus obtained, is in \mathcal{Q} . Hence, by Lemma 3.1, Ω_1^* cannot be extremal for Problem 2 within \mathcal{Q} . Since this implies a contradiction, Γ_1^* contains no nonradial straight line segments in S .

Thus, Γ_1^* contains only radial sides connecting points of ∂K_d to ∂K_M , radial slits, and arcs of ∂K_d and ∂K_M .

We next assume that Γ_1^* contains at least three radial sides, not on slits. From the monotonicity property of $\sigma(w)$, one of the radial sides will act as an l_1 in inequality (2.7) while another of these radial sides will act as an l_2 in (2.7). It is clear that if l_1 is rotated outward, as in Variation II, of §III, about its endpoint on ∂K_M while l_2 is rotated inward, as in Variation I, about its endpoint on ∂K_d , while taking $\phi(w)$ so that equality holds in (2.8), then

starlikeness is preserved and the new domain obtained is in \mathcal{Q} . Since this implies a contradiction, Lemma 5.7 is proved.

LEMMA 5.8. $f^*(z, d, M)$, defined by (1.2), is an extremal function for Problem 2 for the class $\mathcal{S}_d^*(M)$.

PROOF. Let Ω_1^* be the extremal domain of Lemma 5.5 and Lemma 5.7. We show that $\Omega_1^* = e^{i\alpha} f^*(K)$, from which the lemma follows. From the previous lemmas $\Gamma_1^* = \partial\Omega_1^*$ is reduced to having at most two distinct radial sides, not on slits, the connecting arcs of ∂K_d and ∂K_M and at most four radial slits. Thus we need only show that Γ_1^* cannot have any radial slits unless Ω_1^* is a rotation of $K_M \setminus T_1$ where T_1 is the radial slit from $-M$ to $-d$. [Note that in this case $d = \gamma(M)$.] We can assume $|f^*(z)| \leq f^*(re^{i\theta_0})$ where $f^*(z) \in \mathcal{S}_d^*(M)$ and $f^*(K) = \Omega_1^*$. To obtain a contradiction we shall assume Γ_1^* has a radial slit T and Ω_1^* is not a rotation of $K_M \setminus T_1$. Since m.r. $\Omega_1^* = 1$, a routine mapping radius argument shows that either Γ_1^* contains at least two radial slits or $K_M \setminus \overline{\Omega_1^*} \neq \emptyset$. We note, from Lemma 5.6, that if $K_M \setminus \overline{\Omega_1^*} \neq \emptyset$ then Γ_1^* contains at least one straight side. Let m be the point where the minimum value of $\sigma_1(w)$, as defined in (4.5), is attained. We consider two cases: (i) when $m \in T$ and (ii) when $m \notin T$.

For case (i), if there exists a distinct straight side (i.e. not a radial slit) say CD , of Γ_1^* , then CD would act as an l_1 in inequality (2.7) while one of the "sides", say AB , of the slit T , with A at the tip, would act as an l_2 in inequality (2.7). We can now rotate $CD = l_1$ outward, as in Variation II of §III, about an endpoint while hinging side AB at A , as in Variation I', of §IV, and taking $\phi(w)$ so that equation (2.8) holds, which yields a domain with mapping radius one. Also, these variations can be performed in such a way as to preserve starlikeness, and the boundary of the new domain obtained remains in the closure of the annulus S . Thus, by Lemma 3.1, Ω_1^* cannot be an extremal domain for Problem 2 within \mathcal{Q} . This contradicts the assumption that Ω_1^* is extremal.

For case (i), where $m \in T$, if there does not exist a distinct side of Γ_1^* , then there exists another radial slit, which we denote as T' . It is clear from the monotonicity of $\sigma_1(w)$, that "one side", say AB , of T with tip at A , can act as an l_2 in inequality (4.6), while a neighborhood, within Γ_1^* , of the tip, say $f^*(e^{i\theta_0})$, of T' can act as an l_1^* in (4.6); i.e., where l_1^* is the image under $f^*(z)$ of a sufficiently small arc, containing $e^{i\theta_0}$, of $|\zeta| = 1$. Then we can shorten the slit T' , as in Variation III of §IV, and hinge the side AB of slit T , as in Variation I', to produce a new domain in \mathcal{Q} . By Lemma 4.1, Ω_1^* cannot be an extremal domain for Problem 2 within \mathcal{Q} , which implies a contradiction.

For case (ii), where $m \notin T$, it follows that, as w moves around Γ_1^* a sufficiently small neighborhood, within Γ_1^* , of the tip of the slit T , can act as an l_1^* in inequality (4.6), while "one side", say AB , of T , with A at the tip, has

a point C in AB such that the straight line segment CB can act as an l_2 in (4.6). Then we can shorten the slit T and hinge the side AB of the slit T at C to produce a new domain in \mathcal{Q} . Again by Lemma 4.1, this implies a contradiction.

Thus we have proved that Γ_1^* consists of a single arc DC of ∂K_M , two radial sides AD and BC [the sides of a slit when $d = \gamma(M)$] and the arc AB (point) of ∂K_d , i.e., $\Omega_1^* = e^{i\alpha f(K)}$ where $f^*(z) = f^*(z, d, M)$ and Lemma 5.8 is proved.

If we take $\phi(w)$ positive on those parts of the boundary where $\sigma(w)$ [$\sigma_1(w)$] is small and negative where $\sigma(w)$ [$\sigma_1(w)$] is large, i.e., interchanging the roles of l_1 (l_1^*) and l_2 , then an analogous argument proves that the same domain $f^*(K)$ also gives the least minimum modulus for all $f(z)$ in $\mathcal{S}_d^*(M)$.

Thus, the function $f^*(z, d, M)$, as defined by (1.2), is an extremal function giving the sharp upper and lower bounds for $|f(z)|$, within $\mathcal{S}_d^*(M)$, $|z| = r$ fixed, $0 < r < 1$. Since $f^*(z, d, M)$ is a circular symmetric function [6] with respect to the positive real axis, the function, $|f^*(z, d, M)|$, attains its maximum and minimum values for $z = r, -r$ respectively. Therefore, Theorem 5.1 is proved.

THEOREM 5.9. *If $f(z) \in \mathcal{S}_d^*(M)$, $\gamma(M) < d \leq 1$, $1 \leq M < \infty$ and $|z| = r$,*

$$(5.10) \quad 0 < r < \frac{1}{2}(3 - \sqrt{5}),$$

then, for $f^(z) = f^*(z, d, M)$, as defined in §I,*

$$\min_{|z|=r} |f^{*'}(z)| \leq |f'(z)| \leq \max_{|z|=r} |f^{*'}(z)|.$$

PROOF. As in Theorem 5.1 it is only necessary to prove the result for the functional $\operatorname{Re}\{\log f'(z)\}$. We use the method of proof of Theorem 5.1. It is clear that Theorem 5.9 will follow upon showing that the function obtained from $f'(z)$ corresponding to the $\sigma(w)$ in (2.6) and $\sigma_1(w)$ in (4.6) has the same monotonic behavior as $\sigma(w)$ for r satisfying (5.10).

We first proceed as in §II. In this proof, all the integrals are taken over the boundary Γ and $d\Psi$ is defined as in (2.2). We shall assume that z_0 is such that $\operatorname{Re}\{\log f'(z)\} < \operatorname{Re}\{\log f'(z_0)\}$. Thus, using known estimates on functions in \mathcal{S}^* it follows that $A(z_0) = 1 + z_0 f''(z_0)/f'(z_0)$ satisfies

$$(5.11) \quad \frac{1 - 4|z_0| + |z_0|^2}{1 - |z_0|^2} \leq A(z_0) \leq \frac{1 + 4|z_0| + |z_0|^2}{1 - |z_0|^2}.$$

We have from equation (2.1) that

$$(5.12) \quad \hat{f}(z) = f(z) \left[1 + \frac{\varepsilon}{2\pi} \int \frac{zf'(z)}{f(z)} \frac{\xi + z}{\xi - z} d\Psi \right] + o(\varepsilon).$$

Hence, by taking the derivative in (5.12), we obtain

$$\hat{f}'(z) = f'(z) \left\{ 1 + \frac{\varepsilon}{2\pi} \int \left[\frac{2\xi z}{(\xi - z)^2} + \left(1 + \frac{zf''(z)}{f'(z)} \right) \frac{\xi + z}{\xi - z} \right] d\Psi \right\} + o(\varepsilon).$$

So

$$\log \hat{f}'(z) = \log f'(z) + \frac{\varepsilon}{2\pi} \int \left[\frac{2\xi z}{(\xi - z)^2} + A(z) \frac{\xi + z}{\xi - z} \right] d\Psi + o(\varepsilon),$$

where $A(z) = 1 + zf''(z)/f'(z)$. Taking real parts we obtain

$$\operatorname{Re}\{\log \hat{f}'(z)\} - \operatorname{Re}\{\log f'(z)\} = \frac{\varepsilon}{2\pi} \int \lambda(\xi) d\Psi + o(\varepsilon),$$

where

$$(5.13) \quad \lambda(\xi) = \operatorname{Re} \left\{ \frac{2\xi z}{(\xi - z)^2} + A(z) \frac{\xi + z}{\xi - z} \right\}.$$

If we define

$$F(x) = \frac{2x}{(1-x)^2} + A(z_0) \frac{1+x}{1-x} = A + 2(1+A)x + \dots,$$

where $x = \bar{\xi}z_0$ in (5.13), then $F(x)$ defines an analytic function of $x \in K$. Using (5.11) a straightforward calculation shows that $\operatorname{Re}\{1 + xF''(x)/F'(x)\} > 0$ for $0 < |x| = |\bar{\xi}z_0| = r < \frac{1}{2}(3 - \sqrt{5})$. Thus, for r satisfying (5.10), $F(x)$ maps $|x| = r$ onto a convex curve. Consequently $\lambda(\xi)$ has the same monotonic property as $\sigma[w(\xi)]$ in (2.6).

Now we proceed as in §IV. From equation (4.2) we have

$$\begin{aligned} \hat{g}'(z, t, \varepsilon) &= g'(z, t) \left\{ 1 - \frac{\varepsilon}{2\pi} \int \left[\frac{2\xi z}{(\xi - z)^2} + \left(1 + \frac{zg''(z, t)}{g'(z, t)} \right) \frac{\xi + z}{\xi - z} \right] d\Psi \right\} + o(\varepsilon). \end{aligned}$$

Thus

$$\begin{aligned} \operatorname{Re}\{\log \hat{g}'(z, t, \varepsilon)\} &= \operatorname{Re}\{\log g'(z, t)\} \\ &\quad - \frac{\varepsilon}{2\pi} \int \operatorname{Re} \left\{ \frac{2\xi z}{(\xi - z)^2} + \left(1 + \frac{zg''(z, t)}{g'(z, t)} \right) \frac{\xi + z}{\xi - z} \right\} d\Psi + o(\varepsilon) \\ &= \operatorname{Re}\{\log g'(z, t)\} - \frac{\varepsilon}{2\pi} \int \lambda(\xi) d\Psi + o(\varepsilon). \end{aligned}$$

Hence

$$\begin{aligned}
& \frac{\partial}{\partial \varepsilon} \left[\log |\hat{g}'(z, t, \varepsilon)| \right] \Big|_{\varepsilon=0} \\
&= \left\{ \operatorname{Re} \left[\frac{\partial^2 g(z, t)}{\partial \varepsilon \partial z} g'(z, t) \right] - \varepsilon \cdot \frac{\partial}{\partial \varepsilon} \left[\frac{1}{2\pi} \int \lambda(\zeta) d\Psi + o(1) \right] \right. \\
&\quad \left. - \frac{1}{2\pi} \int \lambda(\zeta) d\Psi + o(1) \right\} \Big|_{\varepsilon=0} \\
&= \operatorname{Re} \left\{ \frac{\partial}{\partial z} \frac{\partial g}{\partial t} \cdot \frac{\partial t}{\partial \varepsilon} / g'(z, t) \right\} \Big|_{\varepsilon=0} - \frac{1}{2\pi} \int \lambda(\zeta) d\Psi \\
&= \operatorname{Re} \left\{ \frac{2\zeta_0 z}{(\zeta_0 - z)^2} + A(z) \frac{\zeta_0 + z}{\zeta_0 - z} \right\} \cdot \frac{1}{2\pi} \int d\Psi - \frac{1}{2\pi} \int \lambda(\zeta) d\Psi, \\
&= \lambda(\zeta_0) \frac{1}{2\pi} \int d\Psi - \frac{1}{2\pi} \int \lambda(\zeta) d\Psi,
\end{aligned}$$

where the notation and justifications are as in §IV. Thus, we have produced a $\lambda(\zeta)$ with the required monotonic property as shown above for r satisfying (5.10). Therefore we may continue as in the proof of Theorem 5.1 and Theorem 5.9 follows.

Note. In his dissertation the author, using similar techniques, solved Problems 1 and 2 for the class $C_d(M)$ of convex functions in $\mathbb{S}_d^*(M)$. The boundary of the extremal domain is an arc AC of ∂K_d , two straight sides AB , CD , tangent to ∂K_d at A and C respectively, with B and D either on ∂K_M or $B = D$ (or both) and the arc BD (if it exists) of ∂K_M .

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